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ON THE SOLUTIONS OF SOME SYSTEMS OF LINEAR REAL OCTONION EQUATIONS

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Abstract The problem of finding solutions to a system of linear quaternion or octonion equations arises from many practical applications. This paper studies the systems of linear real octonion equations

$$\begin{matrix} ax+yb=c \\ xa+yb=d \end{matrix}, \quad \begin{matrix} ax+yb=c \\ ax+by=d \end{matrix}, \quad \begin{matrix} ax+ya=c \\ xb+yb=d \end{matrix},$$

$$\begin{matrix} ax+by=c \\ xa+by=d \end{matrix}, \quad \begin{matrix} ax+yb=c \\ xb+yb=d \end{matrix}, \quad \begin{matrix} ax+yb=c \\ xa+by=d \end{matrix}.$$

We present some necessary and sufficient conditions for the existence of solutions to these systems and give expressions of the general solutions to these systems when the solvability conditions are satisfied.

1 Introduction

It is known that linear equations and systems have been one of the main topics in linear and nonlinear algebra theory and its applications. The primary work in the investigation of a linear equation (system) is to give solvability conditions and general solutions to the equation(s). Among other concerns with a linear equation (system) are the uniqueness of solution, minimal norm solutions, least-squares solutions, various symmetric solutions, the maximal and minimal ranks of solutions.

Various kinds of linear quaternion and octonion equations and systems have received much attention in the literature. (see, for example, [1, 2, 6-17] and the references therein). Due to the nonassociative and noncommutativity, one cannot directly extend various results on real and complex numbers to octonions. The book by Conway and Smith [3] gives a great deal of useful background on octonions, much of it based on Coxeter's paper [4]. The linear equation $ax+by=c$ called as the Sylvester equation plays a very vital role in control systems design, such as eigenstructure assignment [8, 12], pole assignment [14, 18] and observer design [5]. In [11] authors have described the set of solutions of the equation $x\alpha = x + \beta$ over an algebraic division ring. The author of the paper [13] has classified solutions of the quaternionic equation $ax+xb=c$. In [16] it is solved the linear equations of the forms $ax = xb$ and $ax = \overline{x}b$ in the real Cayley–Dickson algebras (quaternions, octonions, sedenions), and established a form of roots of such equations. In [6] it is investigated the solutions of the equations of the forms $ax = xb$ and $ax = \overline{x}b$ for some generalizations of quaternions and octonions. In [15], the general linear quaternionic equation with one unknown and systems of linear quaternionic equations with two unknown are solved. In [9], the quaternionic equation $ax + xb = c$ is studied. In [1], Bolat and İpek first have considered the linear octonionic equation with one unknown of the form $\alpha(x\alpha) = (\alpha x)\alpha = \alpha x\alpha = \rho$,with $0 \neq \alpha \in \mathbf{O}$, second, they have presented a method which allows to reduce any octonionic equation with the left and right coefficients to a real system of eight equations and finally reached the solutions of this linear octonionic equation from this real system. In [2], some complex quaternionic equations in the type $ax - xb = c$ are investigated.

Although there is a considerable interest in studying linear quaternion and octonion equations (see, for example, [1, 2, 6, 9, 11, 13, 15, 17], only few papers have studied systems of linear quaternion and octonion equations so far [15, 17].

This paper aims to study the systems of linear real octonion equations

$$\left.\begin{array}{lll} \begin{array}{l} ax+yb=c \\ xa+yb=d \end{array}, & \begin{array}{l} ax+yb=c \\ ax+by=d \end{array}, & \begin{array}{l} ax+ya=c \\ xb+yb=d \end{array}, \\ \begin{array}{l} ax+by=c \\ xa+by=d \end{array}, & \begin{array}{l} ax+yb=c \\ xb+yb=d \end{array}, & \begin{array}{l} ax+yb=c \\ xa+by=d \end{array}. \end{array}\right\} \tag{1.1}$$

over the real number field, discussing solvability conditions and giving explicit expressions of the solutions when these systems is solvable. Some preliminaries about the basic idea of this paper are provided in Section 2. In Section 3, we present some necessary and sufficient conditions for the existence of a solution to these systems and give an expression of the general solution to the systems when the solvability conditions are satisfied. Some conclusions are given in Section 4.

2 Some Preliminaries

In this section, we shortly review some definitions, notations and basic properties which we need to use in the presentations and proofs of our main results.

The octonions in Clifford algebra **C** are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The field $\mathbf{O} \cong \mathbf{C}^4$ of octonions

$$\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3\, e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7, a_i (i = 0, 1, ..., 7) \in \mathbb{R}$$

is an eight-dimensional non-commutative and non-associative $\mathbb{R}$-field generated by eight base elements $e_0, e_1, ..., e_6$ and e_7. The multiplication rules for the basis of **O** are listed in the following table

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table: The multiplication table for the basis of **O.**

The conjugate of α is defined by

$$\overline{\alpha} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3\, e_3 - \alpha_4 e_4 - \alpha_5 e_5 - \alpha_6 e_6 - \alpha_7 e_7$$

and the octonions α and β satisfy $\overline{(\alpha\beta)} = \overline{\beta}\overline{\alpha}$.

The real and the imaginary parts of α are given by

$$\frac{\alpha + \overline{\alpha}}{2} = \alpha_0 e_0$$

and

$$\frac{\alpha - \overline{\alpha}}{2} = \sum_{k=1}^{7} \alpha_k e_k,$$

respectively.

The product of an octonion with its conjugate, $\overline{\alpha}\alpha = \alpha\overline{\alpha}$, is always a nonnegative real number:

$$\overline{\alpha}\alpha = \sum_{k=0}^{7} \alpha_k^2. \tag{2.1}$$

Using this, the norm of an octonion can be defined as

$$\|\alpha\| = \sqrt{\overline{\alpha}\alpha}.$$

This norm agrees with the standard Euclidean norm on $\mathbb{R}^8$ and the octonions α and β satisfy $\|\alpha\beta\| = \|\alpha\| \, \|\beta\|$.

The existence of a norm on **O** implies the existence of inverses for every nonzero element of **O**. The inverse of $\alpha \neq 0$ is given by

$$\alpha^{-1} = \frac{\overline{\alpha}}{\|\alpha\|^2} \tag{2.2}$$

and it satisfies $\alpha^{-1}\alpha = \alpha\alpha^{-1} = 1$.

For $k \in \mathbb{R}$, the octonion $k.\alpha$ is the octonion

$$k.\alpha = \sum_{i=0}^{7} (k\alpha_i)\, e_i. \tag{2.3}$$

The scalar product of the octonions α, $\beta \in \mathbf{O}$ is

$$\langle \alpha, \beta \rangle = \sum_{i=0}^{7} \alpha_i \beta_i. \tag{2.4}$$

Also, although **O** is nonassociative, for all $\alpha, \beta \in \mathbf{O}$, the following equalities hold:

$$\alpha(\alpha\beta) = \alpha^2\beta, \quad (\beta\alpha)\alpha = \beta\alpha^2, \quad (\alpha\beta)\alpha = \alpha(\beta\alpha) = \alpha\beta\alpha. \tag{2.5}$$

A useful method for investigating the problems of equations in octonions is to use real matrix representations. This method has widely been employed for quaternionic equations in some recent papers [16] and [17]. We now present matrix representations of octonions and some basic results related to these representations, which will be serve as a tool for our examination in the sequel.

Definition 2.1. Let $x = \sum_{i=0}^{7} x_i e_i \in \mathbf{O}$. Then $\overrightarrow{x} = [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$ is called as the vector representation of x.

Definition 2.2. [17] Let $\alpha = \alpha' + \alpha'' e \in \mathbf{O}$, where $\alpha' = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\alpha'' = \alpha_4 + \alpha_5 i + \alpha_6 j + \alpha_7 k \in \mathbf{H}$. Then the 8×8 real matrix

$$w(\alpha) = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_5 & \alpha_4 & \alpha_7 & -\alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & -\alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & -\alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_5 & -\alpha_4 & \alpha_7 & -\alpha_6 & \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_6 & -\alpha_7 & -\alpha_4 & \alpha_5 & \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_6 & -\alpha_5 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix} \tag{2.6}$$

is called the left matrix representation of α over $\mathbb{R}$.

Let $c^1_{w(\alpha)}$ be the first column of the matris $w(\alpha)$. Then, it is obviously that $\overrightarrow{\alpha} = c^1_{w(\alpha)}$.

Theorem 2.3. *[17] Let $\alpha,\ x \in \mathbf{O}$ be given. Then*

$$\overrightarrow{\alpha x} = w\left(\alpha\right)\overrightarrow{x}. \tag{2.7}$$

Definition 2.4. [17] Let $\alpha = \alpha' + \alpha'' e \in \mathbf{O}$, where $\alpha' = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\alpha'' = \alpha_4 + \alpha_5 i + \alpha_6 j + \alpha_7 k \in \mathbf{H}$. Then the 8×8 real matrix

$$v\left(\alpha\right) = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 & \alpha_5 & -\alpha_4 & -\alpha_7 & \alpha_6 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 & \alpha_6 & \alpha_7 & -\alpha_4 & -\alpha_5 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 & \alpha_7 & -\alpha_6 & \alpha_5 & -\alpha_4 \\ \alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_5 & \alpha_4 & -\alpha_7 & \alpha_6 & -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_6 & \alpha_7 & \alpha_4 & -\alpha_5 & -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_7 & -\alpha_6 & \alpha_5 & \alpha_4 & -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \tag{2.8}$$

is called the right matrix representation of α over $\mathbb{R}$.

Let $c^1_{v(\alpha)}$ be the first column of the matris $v\left(\alpha\right)$. Then, it is obviously that $\overrightarrow{\alpha} = c^1_{v(\alpha)}$.

Theorem 2.5. *[17] Let $\alpha,\ x \in \mathbf{O}$ be given. Then*

$$\overrightarrow{x\alpha} = v\left(\alpha\right)\overrightarrow{x}. \tag{2.9}$$

Next we give the properties of the obtained left and right real matrix representations for the octonions.

Theorem 2.6. *[17] Let $\alpha,\ x \in \mathbf{O}$, $\lambda \in \mathbb{R}$ be given. Then*

(i) $\alpha = \beta \Leftrightarrow w\left(\alpha\right) = w\left(\beta\right)$,

(ii) $w\left(\alpha+\beta\right) = w\left(\alpha\right) + w\left(\beta\right)$,

(iii) $w\left(\lambda\alpha\right) = \lambda w\left(\alpha\right)$, $w\left(1\right) = I_8$,

(iv) $w\left(\overline{\alpha}\right) = w^T\left(\alpha\right)$,

(v) $\alpha = \beta \Leftrightarrow v\left(\alpha\right) = v\left(\beta\right)$,

(vi) $v\left(\alpha+\beta\right) = v\left(\alpha\right) + v\left(\beta\right)$,

(vii) $v\left(\lambda\alpha\right) = \lambda v\left(\alpha\right)$, $v\left(1\right) = I_8$,

(viii) $v\left(\overline{\alpha}\right) = v^T\left(\alpha\right)$,

(ix) $w^{-1}\left(\alpha\right) = w\left(\alpha^{-1}\right)$, *for* $\alpha \neq 0$,

(x) $v^{-1}\left(\alpha\right) = v\left(\alpha^{-1}\right)$, *for* $\alpha \neq 0$.

Theorem 2.7. *[17] Let $\alpha \in \mathbf{O}$ be given. Then the two matrix representations satisfy the following three identities*

$$w(\alpha^2) = w^2(\alpha),\ \ v(\alpha^2) = v^2(\alpha),\ \ w(\alpha)v(\alpha) = v(\alpha)w(\alpha).$$

Theorem 2.8. *[17] Let $\alpha,\ \beta \in \mathbf{O}$ be given. Then their matrix representations satisfy the following identities*

$$w(\alpha\beta) + w(\beta\alpha) = w(\alpha)w(\beta) + w(\beta)w(\alpha),$$
$$v(\alpha\beta) + v(\beta\alpha) = v(\alpha)v(\beta) + v(\beta)v(\alpha),$$
$$w(\alpha\beta) + v(\alpha\beta) = w(\alpha)w(\beta) + v(\beta)v(\alpha),$$
$$w(\alpha)v(\beta) + w(\beta)v(\alpha) = v(\alpha)w(\beta) + v(\beta)w(\alpha),$$
$$w(\alpha\beta) = w(\alpha)w(\beta) + w(\alpha)v(\beta) - v(\beta)w(\alpha)$$
$$v(\alpha\beta) = v(\beta)v(\alpha) + w(\beta)v(\alpha) - v(\alpha)w(\beta).$$

Also, note that the Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four matrix equations

$$(i)\ AXA = A,\quad (ii) XAX = X,\quad (iii)(AX)^{*} = AX,\quad (iv)(XA)^{*} = XA,$$

and a matrix is called as a g-inverse of A, denoted by $X = A^{-}$, if it satisfies $AXA = A$.

Throughout this paper, let the symbol $\delta(\alpha,\beta)$ stand for the matrix $w(\alpha) - v(\beta)$.

The following theorem gives a formula for the g-inverse of $\delta(\alpha,\alpha)$.

Theorem 2.9. *[17] Let $\alpha \in \mathbf{O}$ be given with $\alpha \notin \mathbb{R}$. Then*

$$\delta^{3}(\alpha,\alpha) = -4\,|Im\alpha|^{2}\,\delta(\alpha,\alpha), \tag{2.10}$$

and $\delta(\alpha,\alpha)$ has a generalized inverse as follows

$$\delta^{-}(\alpha,\alpha) = -\frac{1}{4\,|Im\alpha|^{2}}\delta(\alpha,\alpha). \tag{2.11}$$

Theorem 2.10. *[17] Let $\alpha,\ \beta \in \mathbf{O}$ be given. Then the linear equation $\alpha x = x\beta$ has a nonzero solution if and only if*

$$Re\alpha = Re\beta \text{ and } |Im\alpha| = |Im\beta|\,. \tag{2.12}$$

a) *In that case, if $\beta \neq \overline{\alpha}$, i.e. $Im\alpha + Im\beta \neq 0$, then the general solution of $\alpha x = x\beta$ can be expressed as*

$$x = (Im\alpha)\,p + p\,(Im\beta)\,, \tag{2.13}$$

where $p \in A(\alpha,\beta)$, the subalgebra generated by α and β, is arbitrary, or equivalently

$$x = \lambda_1\,(Im\alpha + Im\beta) + \lambda_2\,[|Im\alpha|\,|Im\beta| - (Im\alpha)\,(Im\beta)]\,, \tag{2.14}$$

where $\lambda_1,\lambda_2 \in \mathbb{R}$ are arbitrary.

b) *If $\beta = \overline{\alpha}$, then the general solution of $\alpha x = x\beta$ is*

$$x = x_1e_1 + x_2e_2 + ... + x_7e_7, \tag{2.15}$$

where $x_1 - x_7$ satisfy $\alpha_1x_1 + \alpha_2x_2 + ... + \alpha_7x_7 = 0$.

In [17], two octonions α and β are called to be similar if there is a nonzero $p \in \mathbf{O}$ such that $\alpha = p\beta p^{-1}$, which is written as $\alpha \sim \beta$. Thus, Theorem 2.10 shows that two octonions are similar if and only if $Re\alpha = Re\beta$ and $|Im\alpha| = |Im\beta|$.

Theorem 2.11. *[17] Let $\alpha,\ \beta \in \mathbf{O}$ be given with $\alpha \notin \mathbb{R}$. Then the linear equation $\alpha x - x\alpha = \beta$ has a solution in $\mathbf{O}$ if and only if the equality $\alpha\beta = \beta\overline{\alpha}$ holds. In this case, the general solution of $\alpha x - x\alpha = \beta$ is*

$$x = \frac{1}{4\,|Im\alpha|^{2}}\,(\beta\alpha - \alpha\beta) + p - \frac{1}{|Im\alpha|^{2}}\,(Im\alpha)\,p\,(Im\alpha)\,, \tag{2.16}$$

where $p \in \mathbf{O}$ is arbitrary.

3 Solving some kinds of two-sided systems of linear equations over O

In this section, we present some necessary and sufficient conditions for the existence of a solution to the systems given in (1.1) and give an expression of the general solution to the systems when the solvability conditions are satisfied.

Proposition 3.1. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{aligned} ax + yb = c \\ xa + yb = d \end{aligned}\right\} \tag{3.1}$$

where $a, b, c, d \in \mathbf{O} - \{0\}$ are given octonions and x, y are unknown octonions. Then the linear equation system (3.1) has a solution in $\mathbf{O}$. In this case, the vector representations of the general solution of system (3.1) are

$$\begin{aligned} \overrightarrow{x} &= \delta^{-}(a,a)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}, \\ \overrightarrow{y} &= v^{-1}(b)\left(\overrightarrow{c} - w(a)\left[\delta^{-}(a,a)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\right]\right), \end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$.

Proof. From the system of linear octonionic equations (3.1) , we obtain

$$ax - xa = c - d. \tag{3.2}$$

According to Eqs. (2.7) and (2.9), the equation (3.2) can equivalently be written as

$$[w(a) - v(a)]\,\overrightarrow{x} = \delta(a,a)\,\overrightarrow{x} = \overrightarrow{c-d}. \tag{3.3}$$

Since $a \sim a$, we know by Theorem 2.11 that $ax = xa$ has a nonzero solution. Thus $\delta(a,a)$ is singular since $a \sim a$. In that case, Eq. (3.2) is solvable if and only if

$$\delta(a,a)\,\delta^{-}(a,a)\,\overrightarrow{c-d} = \overrightarrow{c-d},$$

and in this case the general solution of Eq. (3.3) can be expressed as

$$\overrightarrow{x} = \delta^{-}(a,a)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}, \tag{3.4}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$. From the system of linear octonionic equations (3.1) , the equation $ax + yb = c$, can equivalently be written as

$$w(a)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{c}. \tag{3.5}$$

Then, substituting in the equation (3.5) of the solution $\overrightarrow{x}$, we obtain

$$\overrightarrow{y} = v^{-1}(b)\left(\overrightarrow{c} - w(a)\left[\delta^{-}(a,a)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\right]\right).$$

□

Proposition 3.2. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{aligned} ax + yb &= c \\ ax + by &= d \end{aligned}\right\} \tag{3.6}$$

where $a, b, c, d \in \mathbf{O} - \{0\}$ are given octonions and x, y are unknown octonions. Then the linear equation system (3.6) has a solution in $\mathbf{O}$. In this case, the vector representations of the general solution of system (3.6) are

$$\begin{aligned} \overrightarrow{x} &= w^{-1}(a)\left(\overrightarrow{c} - v(b)\left[\delta^{-}(b,b)\left(\overrightarrow{d-c}\right) + 2\left[I_8 - \delta^{-}(b,b)\,\delta(b,b)\right]\overrightarrow{p}\right]\right), \\ \overrightarrow{y} &= \delta^{-}(b,b)\left(\overrightarrow{d-c}\right) + 2\left[I_8 - \delta^{-}(b,b)\,\delta(b,b)\right]\overrightarrow{p}, \end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$.

Proof. From the system of linear octonionic equations (3.6) , we obtain

$$by - yb = d - c. \tag{3.7}$$

According to Eqs. (2.7) and (2.9), the equation (3.7) can equivalently be written as

$$[w(b) - v(b)]\,\overrightarrow{y} = \delta(b,b)\,\overrightarrow{y} = \overrightarrow{d-c}. \tag{3.8}$$

Since $b \sim b$, we know by Theorem 2.11 that $by = yb$ has a nonzero solution. Thus $\delta(b,b)$ is singular since $b \sim b$. In that case, Eq. (3.7) is solvable if and only if

$$\delta(b,b)\,\delta^{-}(b,b)\,\overrightarrow{d-c} = \overrightarrow{d-c},$$

and in this case the general solution of Eq. (3.8) can be expressed as

$$\overrightarrow{y} = \delta^{-}(b,b)\left(\overrightarrow{d-c}\right) + 2\left[I_8 - \delta^{-}(b,b)\,\delta(b,b)\right]\overrightarrow{p}, \tag{3.9}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$. From the system of linear octonionic equations (3.6), the equation $ax + yb = c$, can equivalently be written as

$$w(a)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{c}. \tag{3.10}$$

Then, substituting in the equation (3.10) of the solution $\overrightarrow{y}$, we obtain

$$\overrightarrow{x} = w^{-1}(a)\left(\overrightarrow{c} - v(b)\left[\delta^{-}(b,b)\left(\overrightarrow{d-c}\right) + 2\left[I_8 - \delta^{-}(b,b)\,\delta(b,b)\right]\overrightarrow{p}\right]\right).$$

□

Proposition 3.3. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{array}{l} ax + ya = c \\ xb + yb = d \end{array}\right\} \tag{3.11}$$

where $a, b, c, d \in \mathbf{O} - \{0\}$ are given octonions and x, y are unknown octonions. Then the linear equation system (3.11) has a solution in $\mathbf{O}$. In this case, the vector representations of the general solution of system (3.11) are

$$\begin{aligned} \overrightarrow{x} &= w^{-1}(a)\left(\overrightarrow{c} - v(b)\left[\delta^{-}(a,a)\,w(a)\,v^{-1}(b)\left(\overrightarrow{d} - v(b)\,w^{-1}(a)\,\overrightarrow{c}\right)\right.\right. \\ &\quad \left.\left. +2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\right]\right), \\ \overrightarrow{y} &= \delta^{-}(a,a)\,w(a)\,v^{-1}(b)\left(\overrightarrow{d} - v(b)\,w^{-1}(a)\,\overrightarrow{c}\right) + 2\left[I_8 - \delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}, \end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$.

Proof. According to Eqs. (2.7) and (2.9), the system of linear octonionic equations (3.6) can equivalently be written as

$$\left.\begin{array}{r} w(a)\overrightarrow{x} + v(a)\overrightarrow{y} = \overrightarrow{c} \\ v(b)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{d} \end{array}\right\}. \tag{3.12}$$

Then, from first equation in here we have

$$w(a)\overrightarrow{x} = \overrightarrow{c} - v(a)\overrightarrow{y}.$$

Since $a \neq 0$, thus $w(a)$ is invertible matrix, and so a direct calculation gives

$$\overrightarrow{x} = w^{-1}(a)\left[\overrightarrow{c} - v(a)\overrightarrow{y}\right].$$

From the $\overrightarrow{x}$ obtained in here and the equation

$$v(b)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{d},$$

it is easily derived that

$$v(b)w^{-1}(a)\left[\overrightarrow{c} - v(a)\overrightarrow{y}\right] + v(b)\overrightarrow{y} = \overrightarrow{d}.$$

With right distributive law in octonions, we can easily obtain

$$-v(b)w^{-1}(a)v(a)\overrightarrow{y}+v(b)\overrightarrow{y}=\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}$$

or

$$v(b)\left[-w^{-1}(a)v(a)+I_8\right]\overrightarrow{y}=\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}.$$

Since $b\neq 0$, thus $v(b)$ is invertible matrix, and therefore, it is easily obtained

$$\left[-w^{-1}(a)v(a)+I_8\right]\overrightarrow{y}=v^{-1}(b)\left[\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}\right].$$

With right distributive law in octonions, the relation in here can be equivalently rewritten as

$$w^{-1}(a)\left[-v(a)+w(a)\right]\overrightarrow{y}=v^{-1}(b)\left[\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}\right].$$

So we can obtain the following conclusion

$$\left[-v(a)+w(a)\right]\overrightarrow{y}=w(a)v^{-1}(b)\left[\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}\right] \tag{3.13}$$

Since $a\sim a$, we know by Theorem 2.11 that $ax=xa$ has a nonzero solution. Thus $\delta(a,a)$ is singular since $a\sim a$. In that case, Eq. (3.13) is solvable if and only if

$$\delta(a,a)\,\delta^{-}(a,a)\,\overrightarrow{e}=\overrightarrow{e}\text{ with }\overrightarrow{e}=w(a)v^{-1}(b)\left[\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}\right],$$

and in this case the general solution of equation (3.13) can be expressed as

$$\overrightarrow{y}=\delta^{-}(a,a)\,w(a)v^{-1}(b)\left[\overrightarrow{d}-v(b)w^{-1}(a)\overrightarrow{c}\right]+2\left[I_8-\delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}, \tag{3.14}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$. Then, substituting in the equation

$$\overrightarrow{x}=w^{-1}(a)\left[\overrightarrow{c}-v(a)\overrightarrow{y}\right]$$

of the solution $\overrightarrow{y}$, we obtain

$$\begin{aligned}\overrightarrow{x} &= w^{-1}(a)\Big(\overrightarrow{c}-v(b)\Big[\delta^{-}(a,a)\,w(a)\,v^{-1}(b)\left(\overrightarrow{d}-v(b)\,w^{-1}(a)\,\overrightarrow{c}\right)\\ &\quad +2\left[I_8-\delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\Big]\Big).\end{aligned}$$

Thus, the proof has been completed. □

Proposition 3.4. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{array}{l} ax+by=c\\ xa+by=d\end{array}\right\} \tag{3.15}$$

where $a,b,c,d\in\mathbf{O}-\{0\}$ are given octonions and x,y are unknown octonions. Then the linear equation system (3.15) has a solution in $\mathbf{O}$. In this case, the vector representations of the general solution of system (3.15) are

$$\begin{aligned}\overrightarrow{x} &= \delta^{-}(a,a)\left(\overrightarrow{c-d}\right)+2\left[I_8-\delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\\ \overrightarrow{y} &= w^{-1}(b)\left(\overrightarrow{c}-w(a)\left[\delta^{-}(a,a)\left(\overrightarrow{c-d}\right)+2\left[I_8-\delta^{-}(a,a)\,\delta(a,a)\right]\overrightarrow{p}\right]\right),\end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$.

Proof. From the system of linear octonionic equations (3.15) , we obtain

$$ax - xa = c - d. \tag{3.16}$$

According to Eqs. (2.7) and (2.9), the equation (3.16) can equivalently be written as

$$[w(a) - v(a)] \overrightarrow{x} = \delta (a, a) \overrightarrow{x} = \overrightarrow{c - d}. \tag{3.17}$$

Since $a \sim a$, we know by Theorem 2.11 that $ax = xa$ has a nonzero solution. Thus $\delta (a, a)$ is singular since $a \sim a$. In that case, Eq. (3.16) is solvable if and only if

$$\delta (a, a) \delta^{-} (a, a) \overrightarrow{c - d} = \overrightarrow{c - d},$$

and in this case the general solution of equation (3.17) can be expressed as

$$\overrightarrow{x} = \delta^{-} (a, a) \left(\overrightarrow{c - d}\right) + 2 \left[I_8 - \delta^{-} (a, a) \delta (a, a)\right] \overrightarrow{p}, \tag{3.18}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$. From the system of linear octonionic equations (3.15) , the equation $ax + by = c$, can equivalently be written as

$$w(a)\overrightarrow{x} + w(b)\overrightarrow{y} = \overrightarrow{c}. \tag{3.19}$$

Then, substituting in the equation (3.19) of the solution $\overrightarrow{x}$, we obtain

$$\overrightarrow{y} = w^{-1} (b) \left(\overrightarrow{c} - w(a) \left[\delta^{-} (a, a) \left(\overrightarrow{c - d}\right) + 2 \left[I_8 - \delta^{-} (a, a) \delta (a, a)\right] \overrightarrow{p}\right]\right).$$

□

Proposition 3.5. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{array}{l} ax + yb = c \\ xb + yb = d \end{array}\right\} \tag{3.20}$$

where $a, b, c, d \in \mathbf{O} - \{0\}$ are given octonions and x, y are unknown octonions. Then the linear equation system (3.20) has a solution in $\mathbf{O}$. If a is similar to b then the vector representations of the general solution of system (3.20) are

$$\begin{aligned} \overrightarrow{x} &= \delta^{-} (a, b) \left(\overrightarrow{d - c}\right) + 2 \left[I_8 - \delta^{-} (a, b) \delta (a, b)\right] \overrightarrow{p}, \\ \overrightarrow{y} &= v^{-1} (b) \left[\overrightarrow{c} - w (a) \left[\delta^{-} (a, b) \left(\overrightarrow{d - c}\right) + 2 \left[I_8 - \delta^{-} (a, b) \delta (a, b)\right] \overrightarrow{p}\right]\right], \end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$, and if a is not similar to b then the vector representations of the solution of system (3.20) are

$$\begin{aligned} \overrightarrow{x} &= \delta^{-1} (a, b) \left(\overrightarrow{c - d}\right), \\ \overrightarrow{y} &= v^{-1} (b) \left[\overrightarrow{c} - w(a) \left(\delta^{-1} (a, b) \left(\overrightarrow{c - d}\right)\right)\right]. \end{aligned}$$

Proof. From the system of linear octonionic equations (3.20) , we obtain

$$ax - xb = c - d. \tag{3.21}$$

According to Eqs. (2.7) and (2.9), the equation (3.21) can equivalently be written as

$$[w(a) - v(b)] \overrightarrow{x} = \delta (a, b) \overrightarrow{x} = \overrightarrow{c - d}. \tag{3.22}$$

Under $a \sim b$, we know by Theorem 2.11 that $ax = xb$ has a nonzero solution. Thus $\delta (a, b)$ is singular under $a \sim b$. In that case, Eq. (3.21) is solvable if and only if

$$\delta (a, b) \delta^{-} (a, b) \overrightarrow{c - d} = \overrightarrow{c - d},$$

and in this case the general solution of equation (3.22) can be expressed as

$$\overrightarrow{x} = \delta^{-}(a,b)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,b)\,\delta(a,b)\right]\overrightarrow{p}, \tag{3.23}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$. From the system of linear octonionic equations (3.20) , the equation $ax + yb = c$, can equivalently be written as

$$w(a)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{c}. \tag{3.24}$$

Then, substituting in the equation (3.24) of the solution $\overrightarrow{x}$, we obtain

$$\overrightarrow{y} = v^{-1}(b)\left(\overrightarrow{c} - w(a)\left[\delta^{-}(a,b)\left(\overrightarrow{c-d}\right) + 2\left[I_8 - \delta^{-}(a,b)\,\delta(a,b)\right]\overrightarrow{p}\right]\right).$$

If a is not similar to b, clearly the equation (3.22) has a unique solution

$$\overrightarrow{x} = \delta^{-1}(a,b)\left(\overrightarrow{c-d}\right).$$

Thus, substituting in the equation (3.24) of the solution $\overrightarrow{x}$, we obtain

$$\overrightarrow{y} = v^{-1}(b)\left[\overrightarrow{c} - w(a)\left(\delta^{-1}(a,b)\left(\overrightarrow{c-d}\right)\right)\right].$$

□

Proposition 3.6. *Consider a system of linear octonionic equations of the form*

$$\left.\begin{array}{r} ax + yb = c \\ xa + by = d \end{array}\right\} \tag{3.25}$$

where $a, b, c, d \in \mathbf{O} - \{0\}$ are given octonions and x, y are unknown octonions. Then the linear equation system (3.25) has a solution in $\mathbf{O}$. If ab is similar to ba then the vector representations of the general solution of system (3.25) are

$$\begin{aligned} \overrightarrow{x} &= w^{-1}(a)\left(\overrightarrow{c} - v(b)\left[\delta^{-}(ab,ba)\,w(a)\left(\overrightarrow{d} - v(a)w^{-1}(a)\overrightarrow{c}\right)\right.\right. \\ &\quad \left.\left. +2\left[I_8 - \delta^{-}(ab,ba)\,\delta(ab,ba)\right]\overrightarrow{p}\right]\right) \\ \overrightarrow{y} &= \delta^{-}(ab,ab)\left(w(a)\overrightarrow{d} - v(a)\overrightarrow{c}\right) + 2\left[I_8 - \delta^{-}(ab,ab)\,\delta(ab,ab)\right]\overrightarrow{p}, \end{aligned}$$

where $\overrightarrow{p}$ is an arbitrary vector in $\mathbf{O}$, and if ab is not similar to ba then the vector representations of the solution of system (3.25) are

$$\begin{aligned} \overrightarrow{x} &= w^{-1}(a)\left(\overrightarrow{c} - v(b)\left[\delta^{-1}(ab,ba)\,w(a)\left(\overrightarrow{d} - v(a)w^{-1}(a)\overrightarrow{c}\right)\right]\right), \\ \overrightarrow{y} &= \delta^{-1}(ab,ba)\,w(a)\left(\overrightarrow{d} - v(a)w^{-1}(a)\overrightarrow{c}\right). \end{aligned}$$

Proof. According to Eqs. (2.7) and (2.9), the system of linear octonionic equations (3.25) can equivalently be written as

$$\left.\begin{array}{r} w(a)\overrightarrow{x} + v(b)\overrightarrow{y} = \overrightarrow{c} \\ v(a)\overrightarrow{x} + w(b)\overrightarrow{y} = \overrightarrow{d} \end{array}\right\}.$$

Then, from first equation in here we have

$$w(a)\overrightarrow{x} = \overrightarrow{c} - v(b)\overrightarrow{y}.$$

Since $a \neq 0$, thus $w(a)$ is invertible matrix, and so a direct calculation gives

$$\overrightarrow{x} = w^{-1}(a)\left(\overrightarrow{c} - v(b)\overrightarrow{y}\right).$$

Substituting the $\overrightarrow{x}$ obtained in here into the equation

$$v(a)\overrightarrow{x}+w(b)\overrightarrow{y}=\overrightarrow{d}$$

it is easily derived that

$$v(a)w^{-1}(a)\left(\overrightarrow{c}-v(b)\overrightarrow{y}\right)+w(b)\overrightarrow{y}=\overrightarrow{d}.$$

With right distributive law in octonions, we can easily obtain

$$-v(a)w^{-1}(a)v(b)\overrightarrow{y}+w(b)\overrightarrow{y}=\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}$$

or

$$w^{-1}(a)\left[-v(a)v(b)+w(a)w(b)\right]\overrightarrow{y}=\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}. \tag{3.26}$$

Using the facts in Theorem 2.8

$$w(ab)+w(ba)=w(a)w(b)+w(b)w(a) \text{ and } w(ab)+v(ab)=w(a)w(b)+v(b)v(a)$$

we get the following:

$$w(ba)-v(ab)=w(b)w(a)-v(b)v(a).$$

Thus, from here and by the invertibility of the matrix $w^{-1}(a)$ we get

$$\left[w(ab)-v(ba)\right]\overrightarrow{y}=w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right). \tag{3.27}$$

Under $ab\sim ba$, we know by Theorem 2.11 that $(ab)x=x(ba)$ has a nonzero solution. Thus $\delta\left(ab,ba\right)$ is singular under $ab\sim ba$. In that case, Eq. (3.27) is solvable if and only if

$$\delta\left(ab,ba\right)\delta^{-}\left(ab,ba\right)\overrightarrow{e}=\overrightarrow{e} \text{ with } \overrightarrow{e}=w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right).$$

Hence, the general solution of equation (3.8) can be expressed as

$$\overrightarrow{y}=\delta^{-}\left(ab,ba\right)w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right)+2\left[I_8-\delta^{-}\left(ab,ba\right)\delta\left(ab,ba\right)\right]\overrightarrow{p}, \tag{3.28}$$

where $\overrightarrow{p}$ is an arbitrary real vector. Then, substituting in the equation

$$\overrightarrow{x}=w^{-1}(a)\left(\overrightarrow{c}-v(b)\overrightarrow{y}\right)$$

of the solution $\overrightarrow{y}$, we obtain

$$\begin{aligned}\overrightarrow{x} &= w^{-1}\left(a\right)\left(\overrightarrow{c}-v(b)\left[\delta^{-}\left(ab,ba\right)w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right)\right.\right.\\ &\quad \left.\left.+2\left[I_8-\delta^{-}\left(ab,ba\right)\delta\left(ab,ba\right)\right]\overrightarrow{p}\right]\right)\end{aligned}$$

If ab is not similar to ba, clearly the equation (3.27) has a unique solution

$$\overrightarrow{y}=\delta^{-1}\left(ab,ba\right)w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right).$$

Then, substituting in the equation

$$\overrightarrow{x}=w^{-1}(a)\left(\overrightarrow{c}-v(b)\overrightarrow{y}\right)$$

of the solution $\overrightarrow{y}$, we obtain

$$\overrightarrow{x}=w^{-1}\left(a\right)\left(\overrightarrow{c}-v(b)\left[\delta^{-1}\left(ab,ba\right)w(a)\left(\overrightarrow{d}-v(a)w^{-1}(a)\overrightarrow{c}\right)\right]\right).$$

□

4 Conclusions

In this paper, a new computational method based on the left and right matrix representations of octonions is used for solving the systems given in (1.1). This method transforms the equation into a matrix equation and the unknown of this equation is a real vector. Solutions are easily acquired by using this matrix equation, which corresponds to a system of linear algebraic equations. Employing the left and right matrix representations to solve systems of linear real octonion equations is very simple and effective.

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