*ORIGINAL PAPER*

# ON HORADAM SEDENIONS

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***Abstract.*** *Formulas and identities involving many well-known special numbers (such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Horadam, and so on) in the algebras play important roles in themselves and in their diverse applications. Various families of recurring numbers have been established by a number of authors in many different ways. This article presents the results of some new research on a new class of recurring integer sedenion numbers that unite the characteristics of the sedenion numbers and the Horadam numbers.*

***Keywords:*** *Horadam numbers, Fibonacci-like numbers, Sedenion algebra.*

## 1. INTRODUCTION AND PRELIMINARIES

The first four (real) algebras formed in the Cayley-Dickson process are the algebras $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. The next one is the 16-dimensional algebra $S$ of (real) sedenions. It is neither a division nor an alternative algebra. Also sedenion algebra is a non-associative, non-commutative, and non-alternative but power associative 16-dimensional Cayley-Dickson algebra over the $\mathbb{R}$. There is well known fact that sedenion numbers play a great role in mathematics and physics. Over the last years it was considered in several papers by algebraists as well as by mathematical physicists [1-5]. We now recall some definitions and elementary properties of the notions needed in subsequent sections.

A sedenion is defined as follows [1]

$$S=\sum_{i=0}^{15} a_i e_i \tag{1.1}$$

where $a_0, a_1, a_2, ..., a_{15}$ are reals. The 16-dimensional algebra $S$ is the one algebra obtained by the Cayley-Dickson process. For the set $\{e_0, e_1, ..., e_{15}\}$, the multiplication table is given as follows [6]:

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**Table 1. The multiplication table for *S*.**

| . | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1 | 1 | -0 | 3 | -2 | 5 | -4 | -7 | 6 | 9 | -8 | -11 | 10 | -13 | 12 | 15 | -14 |
| 2 | 2 | -3 | -0 | 1 | 6 | 7 | -4 | -5 | 10 | 11 | -8 | -9 | -14 | -15 | 12 | 13 |
| 3 | 3 | 2 | -1 | -0 | 7 | -6 | 5 | -4 | 11 | -10 | 9 | -8 | -15 | 14 | -13 | 12 |
| 4 | 4 | -5 | -6 | -7 | -0 | 1 | 2 | 3 | 12 | 13 | 14 | 15 | -8 | -9 | -10 | -11 |
| 5 | 5 | 4 | -7 | 6 | -1 | -0 | -3 | 2 | 13 | -12 | 15 | -14 | 9 | -8 | 11 | -10 |
| 6 | 6 | 7 | 4 | -5 | -2 | 3 | -0 | -1 | 14 | -15 | -12 | 13 | 10 | -11 | -8 | 9 |
| 7 | 7 | -6 | 5 | 4 | -3 | -2 | 1 | -0 | 15 | 14 | -13 | -12 | 11 | 10 | -9 | -8 |
| 8 | 8 | -9 | -10 | -11 | -12 | -13 | -14 | -15 | -0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 9 | 8 | -11 | 10 | -13 | 12 | 15 | -14 | -1 | -0 | -3 | 2 | -5 | 4 | 7 | -6 |
| 10 | 10 | 11 | 8 | -9 | -14 | -15 | 12 | 13 | -2 | 3 | -0 | -1 | -6 | -7 | 4 | 5 |
| 11 | 11 | -10 | 9 | 8 | -15 | 14 | -13 | 12 | -3 | -2 | 1 | -0 | -7 | 6 | -5 | 4 |
| 12 | 12 | 13 | 14 | 15 | 8 | -9 | -10 | -11 | -4 | 5 | 6 | 7 | -0 | -1 | -2 | -3 |
| 13 | 13 | -12 | 15 | -14 | 9 | 8 | 11 | -10 | -5 | -4 | 7 | -6 | 1 | -0 | 3 | -2 |
| 14 | 14 | -15 | -12 | 13 | 10 | -11 | 8 | 9 | -6 | -7 | -4 | 5 | 2 | -3 | -0 | 1 |
| 15 | 15 | 14 | -13 | -12 | 11 | 10 | -9 | 8 | -7 | 6 | -5 | -4 | 3 | 2 | -1 | -0 |

Sedenions are the hypercomplex numbers. The schoolbook multiplication of two sedenion numbers requires performing 256 real multiplications. Cariow and Cariowa [7] derived an algorithm for the fast multiplication of two sedenions.

There has been an increasing interest on the new results on a new class of recurring integer quaternion, octonion and sedenion numbers that unite the characteristics of the quaternion, octonion and sedenion numbers and the recurring integer numbers.

Horadam firstly defined Horadam numbers on $\mathbb{R}$ and then defined Horadam numbers on $\mathbb{C}$ and $\mathbb{H}$ [8,9]. Horadam quaternions are an important step in the development of contemporary the Cayley-Dickson algebra theory. Later, in [10], Halıcı gave a very complete survey about Horadam quaternions. In [11], Karataş and Halıcı defined Horadam octonions by Horadam sequence which is a generalization of second order recurrence relations. Also, they give some fundamental properties and identities related with these sequences. In [12], Halıcı and Karataş gave a new generalization for sequences of dual quaternions and dual octonions. Moreover, they derived some important identities such as Binet formula, generating function, Cassini identity, sum formula and norm formula by their Binet forms. In [13], Cimen and İpek defined the Jacobsthal and Jacobsthal-Lucas octonions over the octonion algebra $\mathbb{O}$. They presented generating functions and Binet formulas for the Jacobsthal and Jacobsthal-Lucas octonions, and derived some identities of Jacobsthal and Jacobsthal-Lucas octonions. The Horadam numbers are very important in mathematics. Their several properties were studied by many mathematicians and they arise in the examination of various areas of science. For $a,b,p,q \in \mathbb{Z}$, Horadam in [14] defined the Horadam numbers by the recursive equation

$$\{w_n(a,b;p,q)\} : w_n = pw_{n-1} + qw_{n-2}\,;\,(n \geq 2) \tag{1.2}$$

where initial conditions are $w_0 = a,\ w_1 = b,\ n \in \mathbb{N}$ .

For special choices of $a,b,p$ and $q$, the recurrence relations (1.2) generates a number of the remarkable numerical sequences that are widely used in mathematics.

For $a=1,\ b=1$, it is obtained generalized Fibonacci numbers:

$$U_n = pU_{n-1} + qU_{n-2}.$$

For $a=2,\ b=p$, it is obtained generalized Lucas numbers:

$$V_n = pV_{n-1} + qV_{n-2}.$$

For $a=0,\ b=1,\ p=1,\ q=1$, it is obtained classical Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}.$$

For $a=2,\ b=1,\ p=1,\ q=1$, it is obtained classical Lucas numbers:

$$L_n = L_{n-1} + L_{n-2}.$$

For $a=0,\ b=1,\ p=2,\ q=1$, it is obtained Pell numbers:

$$P_n = 2P_{n-1} + P_{n-2}.$$

For $a=2,\ b=2,\ p=2,\ q=1$, it is obtained Pell-Lucas numbers:

$$Q_n = 2Q_{n-1} + Q_{n-2}.$$

For $a=0,\ b=1,\ p=1,\ q=2$, it is obtained Jacobsthal numbers:

$$J_n = J_{n-1} + 2J_{n-2}.$$

For $a=2,\ b=1,\ p=1,\ q=2$, it is obtained Jacobsthal-Lucas numbers:

$$j_n = j_{n-1} + 2j_{n-2}.$$

Binet formulas are well known in the Fibonacci-like numbers theory.

These formulas allow all Horadam numbers $w_n$ to be represented by the roots of $t^2 - pt - q = 0$ that is the characteristic equation of (1.2).

The roots of the characteristic equation are

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta = \frac{p-\sqrt{p^2+4q}}{2}. \tag{1.3}$$

Thus Binet's formula for the sequence $w_n$ to be represented by the roots $\alpha$ and $\beta$:

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.4}$$

where $A = b - a\beta$ and $B = b - a\alpha$.

Horadam numbers has properties that are similar to the classical Fibonacci numbers.

The generating function for Horadam numbers is

$$g(t)=\frac{w_0+(w_1-pw_0)t}{1-pt-qt^2}. \tag{1.5}$$

The Cassini identity for Horadam numbers is

$$w_{n+1}w_{n-1}-w_n^2=q^{n-1}\left(pw_0w_1-w_1^2-w_0^2q\right). \tag{1.6}$$

A summation formula for Horadam numbers is

$$\sum_{i=0}^{n} w_i=\frac{w_1-w_0(p-1)+qw_n-w_{n+1}}{1-p-q}. \tag{1.7}$$

Let us consider the Horadam numbers $w_n$ and define the basic recurrence relation for the Horadam Sedenions. The *n*th Horadam sedenion numbers are given by the following recurrent relation:

$$SG_n=\sum_{i=0}^{15} w_{n+1}e_i. \tag{1.8}$$

This paper is devoted to studying the Horadam sedenions. In this study, we introduce and investigate a new class of Horadam sedenion numbers (such as recurrence relations, summation formulas, Binet's formulas, generating functions and norm formula by their Binet forms). By specifying the parameters, the Horadam sedenions $SG_n(a,b,p,q)$ reduce to some well-known ones such as those under the names Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas sedenions. In [15], Bilgici, Tokeser and Unal defined the Fibonacci and Lucas sedenions over the sedenion algebra $S[\mathbb{R}]$. In [16], Çimen and İpek defined the Jacobsthal and Jacobsthal-Lucas sedenions over the sedenion algebra $S$. They, *n*th Jacobsthal sedenion is $SJ_n=\sum_{s=0}^{15} J_{n+s}e_s$ and the *n*th Jacobsthal-Lucas sedenion is $Sj_n=\sum_{s=0}^{15} j_{n+s}e_s$. They presented the generating functions and Binet formulas for the Jacobsthal and Jacobsthal-Lucas sedenions, and derive some identities of Jacobsthal and Jacobsthal-Lucas sedenions.

## 2. HORADAM SEDENIONS

In this section, we first give some properties of the Horadam sedenion numbers, and then investigate Binet formula, generating function, Cassini identity, summation formula and norm value for these numbers.

After some necessary calculations we acquire the following recurrence

$$SG_{n+1}=pSG_n+qSG_{n-1}.$$

Let $SG_n$ and $SM_n$ be two Horadam sedenions such that $SG_n = w_n e_0 + w_{n+1}e_1 + w_{n+2}e_2 + w_{n+3}e_3 + ... + w_{n+15}e_{15}$, and $SM_n = m_n e_0 + m_{n+1}e_1 + m_{n+2}e_2 + ... + m_{n+15}e_{15}$. The scalar and the vector part of Horadam sedenions $SG_n$ and $SM_n$ are denoted by $S_{SG_n} = w_n e_0$, $\overleftrightarrow{V_{SG_n}} = w_{n+1}e_1 + w_{n+2}e_2 + w_{n+3}e_3 + ... + w_{n+15}e_{15}$, $S_{SM_n} = m_n e_0$ and $\overleftrightarrow{V_{SM_n}} = m_{n+1}e_1 + m_{n+2}e_2 + ... + m_{n+15}e_{15}$, respectively. Therefore, the addition, substraction and multiplication of these sedenions directly are obtained, respectively, as following

$$SG_n \pm SM_n = \sum_{s=0}^{15}\left(w_s \pm m_s\right)e_s \tag{2.1}$$

and

$$\begin{aligned} SG_n.SM_n &= S_{SG_n}S_{SM_n} + S_{SG_n}V_{SM_n} + V_{SG_n}S_{SM_n} \\ &\quad - V_{SG_n}\cdot V_{SM_n} + V_{SG_n}\times V_{SM_n}. \end{aligned} \tag{2.2}$$

The conjugate of $SG_n$ is defined by

$$\overline{SG_n} = w_n e_0 - w_{n+1}e_1 - w_{n+2}e_2 - ... - w_{n+15}e_{15}. \tag{2.3}$$

The norm of $SG_n$ is defined by

$$N_{SG_n} = SG_n\overline{SG_n} = w_n^2 + w_{n+1}^2 + w_{n+2}^2 + ... + w_{n+15}^2. \tag{2.4}$$

Let us find some mathematical properties of the Horadam sedenions introduced above. Now we present some identities for the Horadam sedenions.

**Theorem 2.1.** For $n \geq 2$, we have the following identities:

$$SG_n + \overline{SG_n} = 2w_n e_0, \tag{2.5}$$

$$SG_n^2 + SG_n.\overline{SG_n} = 2w_n.SG_n. \tag{2.6}$$

*Proof:* From (1.8) and (2.3), we get

$$\begin{aligned} SG_n + \overline{SG_n} &= \sum_{s=0}^{15} w_{n+s}e_s + w_n e_0 - \sum_{s=1}^{15} w_{n+s}e_s \\ &= 2w_n e_0 \end{aligned}$$

which gives (2.5). On the other hand, from (2.2) and (2.5) we have

$$\begin{aligned} SG_n^2 = SG_n.SG_n &= SG_n\left(2w_n e_0 - \overline{SG_n}\right) \\ &= 2w_n.SG_n - SG_n.\overline{SG_n} \end{aligned}$$

and so

$$SG_n^2 + SG_n.\overline{SG_n} = 2w_n.SG_n.$$

**Theorem 2.2.** For $n \geq 2$, we have the following identities:

$$qSG_n + pSG_{n+1} = SG_{n+2}.$$

*Proof:* It follows from (1.8) and (2.1) that

$$qSG_n + pSG_{n+1} = q\sum_{s=0}^{15} w_{n+s}e_s + p\sum_{s=0}^{15} w_{n+1+s}e_s$$

$$= \sum_{s=0}^{15}(q \cdot w_{n+s} + p \cdot w_{n+1+s})e_s,$$

and therefore by using the identity $w_n = pw_{n-1} + qw_{n-2}$ (12); we get

$$qSG_n + pSG_{n+1} = \sum_{s=0}^{15} w_{n+2+s}e_s$$
$$= SG_{n+2}.$$

We now will give Binet's formulas for Horadam sedenions.

The characteristic equation of the relation $SG_{n+2} = pSG_{n+1} + qSG_n$ is as follows

$$t^2 - pt - q = 0. \tag{2.7}$$

So, the roots of this characteristic equation are $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$. Note that

$$\alpha + \beta = p, \alpha - \beta = \sqrt{p^2 + 4q} \text{ and } \alpha\beta = -q. \tag{2.8}$$

The following theorem gives Binet's formulas for Horadam sedenions the roots of the charactersictic equation associated to the recurrence relation the Eq.(2.7).

**Theorem 2.3.** Binet's formulas for $SG_n$ is as follows

$$SG_n = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta} \tag{2.9}$$

where $A = b - a\beta$, $B = b - a\alpha$, $\underline{\alpha} = \sum_{s=0}^{15} \alpha^s e_s$ and $\underline{\beta} = \sum_{s=0}^{15} \beta^s e_s$.

*Proof:* Using the Binet formula (1.4), we have

$$SG_n = \sum_{i=0}^{15} G_{n+i}e_i$$

$$= \sum_{i=0}^{15} \frac{A\alpha^{n+i} - B\beta^{n+i}}{\alpha - \beta} e_i$$

$$= \sum_{i=0}^{15} \frac{A\alpha^{n+i}}{\alpha - \beta} e_i - \sum_{i=0}^{15} \frac{B\beta^{n+i}}{\alpha - \beta} e_i$$

$$= \frac{A\alpha^n}{\alpha-\beta}\sum_{i=0}^{15}\alpha^i e_i - \frac{B\beta^n}{\alpha-\beta}\sum_{i=0}^{15}\beta^i e_i$$

$$= \frac{A\alpha^n \underline{\alpha} - B\beta^n \underline{\beta}}{\alpha-\beta}.$$

**Theorem 2.4.** The following formulas are true:

$$\sum_{i=1}^{n} SG_i = \left(\frac{B\underline{\beta}\beta^{n+1}}{1-\beta} - \frac{A\underline{\alpha}\alpha^{n+1}}{1-\alpha}\right)\frac{1}{\alpha-\beta} + K, \tag{2.10}$$

where $K$ is as follows,

$$K = \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha}{1-\alpha} - \frac{B\underline{\beta}\beta}{1-\beta}\right).$$

*Proof:* Using the Binet formula for the Horadam sedenions, we can calculate the summation formula as follows

$$\sum_{i=1}^{n} SG_i = \sum_{i=1}^{n}\frac{A\underline{\alpha}\alpha^i - B\underline{\beta}\beta^i}{\alpha-\beta}$$

$$= \frac{A\underline{\alpha}}{\alpha-\beta}\sum_{i=1}^{n}\alpha^i - \frac{B\underline{\beta}}{\alpha-\beta}\sum_{i=1}^{n}\beta^i$$

$$= \frac{A\underline{\alpha}\alpha}{\alpha-\beta}\left(\frac{1-\alpha^n}{1-\alpha}\right) - \frac{B\underline{\beta}\beta}{\alpha-\beta}\left(\frac{1-\beta^n}{1-\beta}\right)$$

$$= \left(\frac{B\underline{\beta}\beta^{n+1}}{1-\beta} - \frac{A\underline{\alpha}\alpha^{n+1}}{1-\alpha}\right)\frac{1}{\alpha-\beta} + \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha}{1-\alpha} - \frac{B\underline{\beta}\beta}{1-\beta}\right).$$

**Theorem 2.5.** The following formulas are true:

$$\sum_{i=1}^{n} SG_{2i} = \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha^{2n+2}}{\alpha^2-1} - \frac{B\underline{\beta}\beta^{2n+2}}{\beta^2-1}\right) + M, \tag{2.11}$$

where $M$ is as follows,

$$M = \frac{1}{\alpha-\beta}\left(\frac{B\underline{\beta}\beta^2}{\beta^2-1} - \frac{A\underline{\alpha}\alpha^2}{\alpha^2-1}\right).$$

*Proof:* Using the Binet formula for $SG_n$, we can calculate the summation formula as follows

$$\sum_{i=1}^{n} SG_{2i} = \sum_{i=1}^{n} \frac{A\underline{\alpha}\alpha^{2i} - B\underline{\beta}\beta^{2i}}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\left[ A\underline{\alpha}\alpha^{2}\sum_{i=1}^{n}\alpha^{2(i-1)} - B\underline{\beta}\beta^{2}\sum_{i=1}^{n}\beta^{2(i-1)} \right]$$

$$= \frac{A\underline{\alpha}\alpha^{2}}{\alpha-\beta}\left(\frac{\alpha^{2n}-1}{\alpha^{2}-1}\right) - \frac{B\underline{\beta}\beta^{2}}{\alpha-\beta}\left(\frac{\beta^{2n}-1}{\beta^{2}-1}\right)$$

$$= \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha^{2n+2}}{\alpha^{2}-1} - \frac{B\underline{\beta}\beta^{2n+2}}{\beta^{2}-1}\right)$$

$$+ \frac{1}{\alpha-\beta}\left(\frac{B\underline{\beta}\beta^{2}}{\beta^{2}-1} - \frac{A\underline{\alpha}\alpha^{2}}{\alpha^{2}-1}\right).$$

**Theorem 2.6**. The following formulas are true:

$$\sum_{i=1}^{n} SG_{2i-1} = \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha^{2n+1}}{\alpha^{2}-1} - \frac{B\underline{\beta}\beta^{2n+1}}{\beta^{2}-1}\right) + L, \tag{2.12}$$

where $L$ is as follows,

$$L = \frac{1}{\alpha-\beta}\left(\frac{B\underline{\beta}\beta}{\beta^{2}-1} - \frac{A\underline{\alpha}\alpha}{\alpha^{2}-1}\right).$$

*Proof:* Using the Binet formula for $SG_n$, we can calculate the summation formula as follows

$$\sum_{i=1}^{n} SG_{2i-1} = \sum_{i=1}^{n} \frac{A\underline{\alpha}\alpha^{2i-1} - B\underline{\beta}\beta^{2i-1}}{\alpha-\beta}$$

$$= \sum_{i=1}^{n} \frac{A\underline{\alpha}\alpha^{2i-1}}{\alpha-\beta} - \sum_{i=1}^{n} \frac{B\underline{\beta}\beta^{2i-1}}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\left[ A\underline{\alpha}\alpha\sum_{i=1}^{n}\alpha^{2(i-1)} - B\underline{\beta}\beta\sum_{i=1}^{n}\beta^{2(i-1)} \right]$$

$$= \frac{A\underline{\alpha}\alpha}{\alpha-\beta}\left(\frac{\alpha^{2n}-1}{\alpha^{2}-1}\right) - \frac{B\underline{\beta}\beta}{\alpha-\beta}\left(\frac{\beta^{2n}-1}{\beta^{2}-1}\right)$$

$$= \frac{1}{\alpha-\beta}\left(\frac{A\underline{\alpha}\alpha^{2n+1}}{\alpha^{2}-1} - \frac{B\underline{\beta}\beta^{2n+1}}{\beta^{2}-1}\right)$$

$$+ \frac{1}{\alpha-\beta}\left(\frac{B\underline{\beta}\beta}{\beta^{2}-1} - \frac{A\underline{\alpha}\alpha}{\alpha^{2}-1}\right).$$

Thus, we complete the proof.

In the following theorem, we state to different Cassini identities which occur from non-commutativity of sedenion multiplication.

**Theorem 2.7.**(*Cassini's identities*) For Horadam sedenions the following identities are hold:

$$SG_{n+1}.SG_{n-1}-SG_n^2=(-q)^{n-1}AB\left[\frac{\beta\underline{\beta}\underline{\alpha}-\alpha\underline{\alpha}\underline{\beta}}{(\alpha-\beta)}\right] \tag{2.13}$$

$$SG_{n-1}.SG_{n+1}-SG_n^2=(-q)^{n-1}AB\left[\frac{\beta\underline{\alpha}\underline{\beta}-\alpha\underline{\beta}\underline{\alpha}}{(\alpha-\beta)}\right] \tag{2.14}$$

*Proof:* Using the Binet's formula in equation (2.13); we get

$$SG_{n+1}.SG_{n-1}-SG_n^2=\left(\frac{A\underline{\alpha}\alpha^{n+1}-B\underline{\beta}\beta^{n+1}}{\alpha-\beta}\right)$$

$$\left(\frac{A\underline{\alpha}\alpha^{n-1}-B\underline{\beta}\beta^{n-1}}{\alpha-\beta}\right)$$

$$-\left(\frac{A\underline{\alpha}\alpha^{n}-B\underline{\beta}\beta^{n}}{\alpha-\beta}\right)^2.$$

If necessary calculations are made, we obtain

$$(-q)^{n-1}AB\left[\frac{\beta\underline{\beta}\underline{\alpha}-\alpha\underline{\alpha}\underline{\beta}}{(\alpha-\beta)}\right].$$

In a similar way, using the Binet's formula in equation (2.14); we obtain

$$SG_{n-1}SG_{n+1}-SG_n^2=\left(\frac{A\underline{\alpha}\alpha^{n-1}-B\underline{\beta}\beta^{n-1}}{\alpha-\beta}\right)$$

$$.\left(\frac{A\underline{\alpha}\alpha^{n+1}-B\underline{\beta}\beta^{n+1}}{\alpha-\beta}\right)$$

$$-\left(\frac{A\underline{\alpha}\alpha^{n}-B\underline{\beta}\beta^{n}}{\alpha-\beta}\right)^2$$

$$=(-q)^{n-1}AB\left[\frac{\beta\underline{\alpha}\underline{\beta}-\alpha\underline{\beta}\underline{\alpha}}{(\alpha-\beta)}\right].$$

which is desired. Thus, the identities are proved.

We now derive the ordinary generating function $\Im(x)=\sum_{n=0}^{\infty}SG_n x^n$ defined by (1.8):

**Theorem 2.8.** For $SG_n$ defined by (1.8), the following is its ordinary generating function:

$$\Im(x)=\frac{SG_0+\left(-pSG_0+SG_1\right)x}{1-px-qx^2}. \tag{2.15}$$

*Proof:* Firstly, we need to write generating function for Horadam sedenions:

$$\Im(x)=SG_0x^0+SG_1x+SG_2x^2+...+SG_nx^n+...$$

Secondly, we need to calculate $px\Im(x)$ and $qx^2\Im(x)$ as the following equations:

$$px\Im(x)=\sum_{n=0}^{\infty}pSG_nx^{n+1} \quad and \quad qx^2\Im(x)=\sum_{n=0}^{\infty}qSG_nx^{n+2}.$$

Finally, if we made necessary calculations, then we have

$$\Im(x)=\frac{SG_0+\left(-pSG_0+SG_1\right)x}{1-px-qx^2}$$

which is the generating function for Horadam sedenions.

**Theorem 2.9.** The norm of *n*th Horadam sedenion is

$$N(SG_n)=\frac{A^2\alpha^{2n}\left(1+\alpha^2+\alpha^4+...+\alpha^{30}\right)}{\left(\alpha-\beta\right)^2}$$

$$+\frac{B^2\beta^{2n}\left(1+\beta^2+\beta^4+...+\beta^{30}\right)}{\left(\alpha-\beta\right)^2}-T$$

where $T$ is

$$T=\frac{2AB\left(-q\right)^n\left(1+\left(-q\right)+\left(-q\right)^2+...+\left(-q\right)^{30}\right)}{\left(\alpha-\beta\right)^2}.$$

*Proof:* The norm of *n*th Horadam sedenion is $N\left(SG_n\right)=SG_n\overline{SG_n}=\overline{SG_n}SG_n=w_n^2+w_{n+1}^2+...+w_{n+15}^2$. Making necessary calculations and using the equalities $w_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta}$, $\alpha+\beta=p$ and $\alpha\beta=-q$, we obtain

$$N(SG_n) = \frac{A^2\alpha^{2n}\left(1+\alpha^2+\alpha^4+...+\alpha^{30}\right)}{\left(\alpha-\beta\right)^2}$$

$$+\frac{B^2\beta^{2n}\left(1+\beta^2+\beta^4+...+\beta^{30}\right)}{\left(\alpha-\beta\right)^2}$$

$$-\frac{2AB\left(-q\right)^n\left(1+\left(-q\right)+\left(-q\right)^2+...+\left(-q\right)^{30}\right)}{\left(\alpha-\beta\right)^2}.$$

## 3. CONCLUSIONS

In this paper, we give a systematic investigation of new classes of sedenion numbers associated with the familiar Horadam numbers. We obtain various results including recurrence relations, summation formulas, Binet's formulas and generating functions for these classes of sedenion numbers. It is interesting to mention here that, whenever a generalized Fibonacci type sedenion number reduces to Fibonacci sedenion number and other related sedenion numbers, the results become relatively more important from the application viewpoint. Therefore, the numerous numbers involving extensions and generalizations of the Fibonacci type numbers are capable of playing important roles in mathematics. Considering the fundamental role of the sedenion numbers and Horadam numbers in the mathematical tools of the modern science, it is possible to suppose that the new theory of the Horadam sedenions will bring the new results.

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